

Hamiltonicity of the random geometric graph

Michael Krivelevich^{*,†} Tobias Müller^{*,‡}

June 14, 2009

Abstract

Let X_1, \dots, X_n be independent, uniformly random points from $[0, 1]^2$. We prove that if we add edges between these points one by one by order of increasing edge length then, with probability tending to 1 as the number of points n tends to ∞ , the resulting graph gets its first Hamilton cycle at exactly the same time it loses its last vertex of degree less than two. This answers an open question of Penrose and provides an analogue for the random geometric graph of a celebrated result of Ajtai, Komlós and Szemerédi and independently of Bollobás on the usual random graph. We are also able to deduce very precise information on the limiting probability that the random geometric graph is Hamiltonian analogous to a result of Komlós and Szemerédi on the usual random graph. The proof generalizes to uniform random points on the d -dimensional hypercube where the edge-lengths are measured using the l_p -norm for some $1 < p \leq \infty$. The proof can also be adapted to show that, with probability tending to 1 as the number of points n tends to ∞ , there are cycles of all lengths between 3 and n at the moment the graph loses its last vertex of degree less than two.

Keywords: random geometric graph, Hamilton cycles.

1 Introduction and statement of result

Let $X_1, X_2, \dots \in [0, 1]^2$ be a sequence of random points, chosen independently and uniformly at random from $[0, 1]^2$. For $n \in \mathbb{N}$ and $r \geq 0$ the *random geometric graph* $G(n, r)$ has vertex set $V_n := \{X_1, \dots, X_n\}$ and an edge $X_i X_j \in E_n$ iff $\|X_i - X_j\| \leq r$. The "hitting radius" $\rho_n(\mathcal{P})$ of an increasing graph property \mathcal{P} is the least r such that $G(n, r)$ satisfies \mathcal{P} , i.e.:

$$\rho_n(\mathcal{P}) := \min\{r \geq 0 : G(n, r) \text{ satisfies } \mathcal{P}\}.$$

Recall that a graph is *Hamiltonian* if it has a Hamilton cycle (that is, a cycle that goes through all the vertices of the graph). An obvious necessary (but not sufficient) condition for the existence of a Hamilton cycle is that the minimum degree is at least two. In this paper we prove the following result:

Theorem 1. $\mathbb{P}[\rho_n(\text{minimum degree} \geq 2) = \rho_n(\text{Hamiltonian})] \rightarrow 1$ as $n \rightarrow \infty$.

This answers a question of Penrose (see [12], page 317) and provides an analogue for the random geometric graph of a celebrated result of Ajtai, Komlós and Szemerédi [1] and independently of Bollobás [3] on the usual random graph. Theorem 1 can be stated alternatively as saying that if we add the edges between the points X_1, \dots, X_n by order of increasing edge length then, with probability tending to 1 as $n \rightarrow \infty$, the resulting graph obtains its first Hamilton cycle at exactly the same time it loses its last vertex of degree < 2 . By combining Theorem 1 with Theorem 8.4 from [12] (for completeness we have repeated the relevant special case of this theorem as Theorem 4 below) we see that:

^{*}School of Mathematical Sciences, Tel Aviv University, Ramat Aviv 69978, Israel. E-mail: krivelev@post.tau.ac.il, tobias@post.tau.ac.il.

[†]Research supported in part by USA-Israel BSF grant 2006322, by grant 1063/08 from the Israel Science Foundation, and by a Pazy memorial award.

[‡]Research partially supported through an ERC advanced grant.

Corollary 2. Let $(r_n)_n$ be a sequence of nonnegative numbers, and write $x_n := \pi n r_n^2 - (\ln n + \ln \ln n)$. Then:

$$\lim_{n \rightarrow \infty} \mathbb{P}[G(n, r_n) \text{ is Hamiltonian}] = \begin{cases} 0 & \text{if } x_n \rightarrow -\infty; \\ \exp [-(\sqrt{\pi} + e^{-x/2})e^{-x/2}] & \text{if } x_n \rightarrow x \in \mathbb{R}; \\ 1 & \text{if } x_n \rightarrow +\infty. \end{cases}$$

Corollary 2 provides an analogue for the random geometric graph of a result by Komlós and Szemerédi [8] on the limiting probability that the usual random graph is Hamiltonian.

Previously, Petit [14] showed that if $(r_n)_n$ is chosen such that $r_n/\sqrt{\ln n/\pi n} \rightarrow \infty$ then the random geometric graph $G(n, r_n)$ is Hamiltonian with probability tending to 1. This was later sharpened by Diaz, Mitsche and Pérez [5] who showed that the same is true whenever $r_n \geq (1 + \varepsilon)\sqrt{\ln n/\pi n}$ with $\varepsilon > 0$ arbitrary (but fixed). Our results are again an improvement and in a sense the final word on Hamiltonicity of the random geometric graph. In Section 4 we shall nonetheless offer an idea for future research on Hamilton cycles in the random geometric graph.

Since writing this paper it has come to our attention that both Balogh, Bollobás and Walters [2] and Pérez and Wormald [13] have independently obtained essentially the same results at pretty much the same time. Earlier Balogh, Kaul and Martin [10] had proved Theorem 1 in the case when the Euclidean norm in the definition of the random geometric graph is replaced by the l_∞ -norm (i.e. we add an edge between two points if their l_∞ -distance is less than r).

Our proof readily extends to arbitrary dimension and the l_p -norm for any $1 < p \leq \infty$ (i.e. the case where the points are i.i.d. uniform on the d -dimensional unit hypercube and $\|\cdot\|$ in the definition of the random geometric graph is the l_p -norm), but we have chosen to focus on the two-dimensional random geometric graph with the Euclidean norm for the sake of the clarity of our exposition. In Section 3 we briefly explain the changes needed to make the proof work in the case of arbitrary dimension and the l_p -norm.

Our proof can also be adapted to show that, with probability tending to 1 as the number of points n tends to infinity, the random geometric graph becomes *pancyclic* (i.e. there are cycles of all lengths between 3 and n) at precisely the same moment it first achieves minimum degree at least two. In Section 4 we give a brief sketch the adaptations needed to squeeze this out of our proof.

Our proof of Theorem 1 is inspired by the analysis in [5]. Let us briefly outline the main steps in the proof. We pick an r that is close to, but slightly less than ρ_n (minimum degree ≥ 2) and we dissect the unit square into squares of side ηr for a small constant η . Next we consider an auxiliary graph \mathcal{D} consisting of the lower left hand corners of those squares of our dissection that have at least 100 of the X_i s in them, where we connect two points of \mathcal{D} if their distance is less than $r' := r(1 - \eta\sqrt{2})$. As it turns out, this auxiliary graph consists of one “giant” component and a number of small components, that are cliques and are very far apart from each other. Moreover, all of the X_i s are within distance r of all the ≥ 100 points in some square of the auxiliary graph, except for a few clusters of “bad” points. These bad clusters form cliques in the underlying random geometric graph, and these cliques are far apart. We now construct a spanning tree \mathcal{T} of the giant component of \mathcal{D} that has maximum degree at most 26. We increase r to $\rho > r$ which is large enough for the random geometric graph to have minimum degree at least two, and we construct the Hamilton cycle while performing a closed walk on \mathcal{T} that traverses every edge of \mathcal{T} exactly twice (once in each direction). Each time the walk visits a node of \mathcal{T} , the cycle visits a fresh X_i inside the corresponding square. While doing this we are able to make small “excursions” to eat up the X_i s in squares belonging to non-giant components of \mathcal{D} , the bad clusters and all the other X_i s.

2 The proof

Recall that a graph $G = (V, E)$ is k -connected if $|V| > k$ and $G \setminus S$ is connected for all sets $S \subseteq V$ of cardinality $|S| < k$. Clearly, having minimum degree at least k is a necessary condition for

k -connectedness, and 2-connectedness is a necessary condition for Hamiltonicity. In our proof of Theorem 1 we shall rely on the following result of Penrose:

Theorem 3 ([11]). *For any (fixed) $k \in \mathbb{N}$ it holds that:*

$$\mathbb{P}[\rho_n(\text{minimum degree} \geq k) = \rho_n(k\text{-connected})] \rightarrow 1,$$

as $n \rightarrow \infty$.

Thanks to this last theorem, it suffices for us to show that $\mathbb{P}[\rho_n(\text{Hamiltonian}) = \rho_n(2\text{-connected})] \rightarrow 1$ in order to prove Theorem 1. We shall also make use of another result of Penrose. The following theorem is a reformulation of a special case of Theorem 8.4 from [12].

Theorem 4. *Let $(r_n)_n$ be a sequence of nonnegative numbers, and write $x_n := \pi n r^2 - (\ln n + \ln \ln n)$. Then:*

$$\lim_{n \rightarrow \infty} \mathbb{P}[G(n, r_n) \text{ has minimum degree} \geq 2] = \begin{cases} 0 & \text{if } x_n \rightarrow -\infty; \\ \exp[-(\sqrt{\pi} + e^{-x/2})e^{-x/2}] & \text{if } x_n \rightarrow x \in \mathbb{R}; \\ 1 & \text{if } x_n \rightarrow +\infty. \end{cases}$$

For $V \subseteq \mathbb{R}^2$ and $r \geq 0$ we shall denote by $G(V, r)$ the (non-random) geometric graph with vertex set V and an edge $vw \in E(G(V, r))$ iff $\|v - w\| \leq r$. The (non-random) geometric graphs $G(V, r)$ have been the subject of considerable research effort and they are often also called *unit disk graphs*.

For $0 < \eta < 1/\sqrt{2}$ and $r > 0$ let $\mathcal{H}_\eta(r)$ denote the unit disk graph $G(P_{\eta r}, r')$ with vertex set $P_{\eta r} := [0, 1]^2 \cap (\eta r)\mathbb{Z}^2$ (that is, $P_{\eta r}$ is the set of all points in $[0, 1]^2$ whose coordinates are integer multiples of ηr) and threshold distance $r' := r(1 - \eta\sqrt{2})$.

Now suppose that we are also given an arbitrary set $V \subseteq [0, 1]^2$ of points. We shall call a vertex $p \in \mathcal{H}_\eta(r)$ *dense* with respect to V if the square $p + [0, \eta r]^2$ contains at least 100 points of V . If a vertex is not dense we will call it *sparse*. If all neighbours of p in $\mathcal{H}_\eta(r)$ are sparse (i.e. if q is sparse for all $q \in B(p, r') \cap \mathcal{H}_\eta(r)$) then we shall say that p is *bad*.

Let $\mathcal{D}_\eta(V, r)$ denote the subgraph of $\mathcal{H}_\eta(r)$ induced by the dense points, and let $\mathcal{B}_\eta(V, r)$ denote the subgraph induced by the bad points. Part of the proof of Theorem 1 will be to show that if $V = \{X_1, \dots, X_n\}$ and r is chosen close to, but slightly smaller than, $\rho_n(2\text{-connected})$ then $\mathcal{H}_\eta(r)$, $\mathcal{D}_\eta(V, r)$ and $\mathcal{B}_\eta(V, r)$ have a number of desirable properties (with probability tending to 1). This will then allow us to finish the proof of our main theorem by purely deterministic arguments. Here is a list of these desirable properties (here and throughout the rest of the paper “component” will always mean a connected component, and diameter will always refer to the geometric diameter of a point set as opposed to the graph diameter):

- (P1) If \mathcal{K} is a component of $\mathcal{D}_\eta(V, r)$ then it either has (geometric) diameter $\text{diam}(\mathcal{K}) < r'$ or $\text{diam}(\mathcal{K}) > 1000r$;
- (P2) If $\mathcal{K}_1, \mathcal{K}_2$ are two distinct components of $\mathcal{D}_\eta(V, r)$ with (geometric) diameters $\text{diam}(\mathcal{K}_1), \text{diam}(\mathcal{K}_2) < r'$ and $p_1 \in \mathcal{K}_1, p_2 \in \mathcal{K}_2$ then $\|p_1 - p_2\| > 1000r$;
- (P3) If $p_1 \in \mathcal{K}$ for some component \mathcal{K} of $\mathcal{D}_\eta(V, r)$ with $\text{diam}(\mathcal{K}) < r'$ and $p_2 \in \mathcal{B}_\eta(V, r)$ is bad then $\|p_1 - p_2\| > 1000r$;
- (P4) If $p, q \in \mathcal{B}_\eta(V, r)$ are bad, then either $\|p - q\| < r'$ or $\|p - q\| > 1000r$;
- (P5) If $p_1, p_2 \in \mathcal{D}_\eta(V, r)$ and $\|p_1 - p_2\| < 25r$ and neither of p_1 or p_2 lies in a component of (geometric) diameter $< r'$ then there is a $p_1 p_2$ -path in $\mathcal{D}_\eta(V, r)$ that stays inside $B(p_1, 100r)$;
- (P6) $\mathcal{D}_\eta(V, r)$ has exactly one component \mathcal{K} of (geometric) diameter $\text{diam}(\mathcal{K}) \geq r'$.

We will say that a sequence of events $(A_n)_n$ holds *with high probability* (w.h.p.) if $\mathbb{P}(A_n) \rightarrow 1$ as $n \rightarrow \infty$. The following proposition takes care of the probabilistic part of the proof of Theorem 1:

Proposition 5. Set $r_n := \sqrt{\ln n / \pi n}$ and $V_n := \{X_1, \dots, X_n\}$. If $\eta > 0$ is sufficiently small (but fixed) then $\mathcal{H}_\eta(r_n)$, $\mathcal{D}_\eta(V_n, r_n)$ and $\mathcal{B}_\eta(V_n, r_n)$ satisfy properties **(P1)**–**(P6)** w.h.p.

Together with the following deterministic result and the two mentioned results by Penrose, Proposition 5 gives Theorem 1.

Theorem 6. Suppose that $0 < \eta < 1/\sqrt{2}$, $V \subseteq \mathbb{R}^2$ and $r > 0$ are such that $\mathcal{D}_\eta(V, r)$ and $\mathcal{B}_\eta(V, r)$ satisfy **(P1)**–**(P6)**, and that $r \leq \rho \leq 2r$ is such that $G(V, \rho)$ is 2-connected. Then $G(V, \rho)$ is also Hamiltonian.

We postpone the proofs of Proposition 5 and Theorem 6 and we first briefly explain how they imply Theorem 1.

Proof of Theorem 1: Let us write $r_n := \sqrt{\ln n / \pi n}$ and $\sigma_n := \rho_n(2\text{-connected})$. By Theorem 3 and the fact that 2-connectedness is a necessary condition for Hamiltonicity it suffices to show that $G(n, \sigma_n)$ is Hamiltonian with high probability. Theorem 3 together with Theorem 4 show that, with high probability, $G(n, r_n)$ is not 2-connected and $G(n, 2r_n)$ is 2-connected. In other words, $r_n < \sigma_n \leq 2r_n$ with high probability. By Proposition 5 we can fix an $\eta \in (0, 1/\sqrt{2})$ such that properties **(P1)**–**(P6)** hold for $\mathcal{H}_\eta(r_n)$, $\mathcal{D}_\eta(V_n, r_n)$, $\mathcal{B}_\eta(V_n, r_n)$ with high probability, where $V_n := \{X_1, \dots, X_n\}$. Thus, with high probability, Theorem 6 applies to η , $V = \{X_1, \dots, X_n\}$, $r = r_n$, $\rho = \sigma_n$, and $G(n, \sigma_n)$ is indeed Hamiltonian with high probability. ■

Our next step is to prove Proposition 5. We will say that a point or set is *within s of the sides* (of $[0, 1]^2$) if it is contained in

$$\text{side}(s) := \{z \in [0, 1]^2 : z_x \in [0, s) \cap (1-s, 1] \text{ or } z_y \in [0, s) \cap (1-s, 1]\},$$

and we will say it is *within s of the corners* (of $[0, 1]^2$) if it is contained in

$$\text{corner}(s) := \{z \in [0, 1]^2 : z_x \in [0, s) \cap (1-s, 1] \text{ and } z_y \in [0, s) \cap (1-s, 1]\}.$$

The following lemma provides an observation that is pivotal in the proof of Proposition 5.

Lemma 7. Set $r_n := \sqrt{\ln n / \pi n}$ and $V_n := \{X_1, \dots, X_n\}$. For every $\varepsilon > 0$ there exists an $\eta_0 = \eta_0(\varepsilon) > 0$ such that for any fixed $0 < \eta < \eta_0$ the following statements hold w.h.p.:

- (i) For every $\mathcal{S} \subseteq \mathcal{H}_\eta(r_n)$ with $|\mathcal{S}| > (1+\varepsilon)\pi\eta^{-2}$ and $\text{diam}(\mathcal{S}) < 10^5 r_n$, there exists a $q \in \mathcal{S}$ that is dense wrt. V_n ;
- (ii) For every $\mathcal{S} \subseteq \mathcal{H}_\eta(r_n) \cap \text{side}(10^5 r_n)$ with $|\mathcal{S}| > (1+\varepsilon)\frac{\pi}{2}\eta^{-2}$ and $\text{diam}(\mathcal{S}) < 10^5 r_n$, there exists a $q \in \mathcal{S}$ that is dense wrt. V_n ;
- (iii) For every $\mathcal{S} \subseteq \mathcal{H}_\eta(r_n) \cap \text{corner}(10^5 r_n)$ with $|\mathcal{S}| > \varepsilon\eta^{-2}$ and $\text{diam}(\mathcal{S}) < 10^5 r_n$, there exists a $q \in \mathcal{S}$ that is dense wrt. V_n .

In the proof of Lemma 7 we shall make use of the following incarnation of the Chernoff-Hoeffding bound. A proof can for instance be found in [12], on page 16.

Lemma 8. Let Z be a $\text{Bi}(n, p)$ -distributed random variable, and $k \leq \mu := np$. Then

$$\mathbb{P}(Z \leq k) \leq \exp[-\mu H(k/\mu)],$$

where $H(x) := x \ln x - x + 1$.

Proof of Lemma 7: Let us choose $\eta_0 := \varepsilon/10^6$ and fix an arbitrary $0 < \eta < \eta_0$. Our choice of η_0 guarantees that $4[2 \cdot 10^5/\eta] < \varepsilon\eta^{-2}/2$ (we can assume w.l.o.g. that $\varepsilon < 1$).

Let \mathcal{U} denote the collection of all $\mathcal{S} \subseteq \mathcal{H}_\eta(r_n)$ that satisfy the conditions for part (i). Let us first count the number of sets in \mathcal{S} . To this end, observe that if $p \in \mathcal{S}$ and $\text{diam}(\mathcal{S}) < 10^5 r_n$ then $\mathcal{S} \subseteq p + (-10^5 r_n, 10^5 r_n)^2$. Notice that

$$|\mathcal{H}_\eta(r_n) \cap (p + (-10^5 r_n, 10^5 r_n)^2)| \leq [2 \cdot 10^5/\eta]^2,$$

for any $p \in \mathbb{R}^2$. This shows that if $N(p)$ denotes the number of $\mathcal{S} \in \mathcal{U}$ that contain p , then

$$N(p) \leq 2^{\lceil 2 \cdot 10^5 / \eta \rceil^2}, \quad (1)$$

and, since this constant upper bound on $N(p)$ holds for all $p \in \mathcal{H}_\eta(r_n)$, it follows that

$$|\mathcal{U}| \leq \sum_{p \in \mathcal{H}_\eta(r_n)} 2^{N(p)} = O(|\mathcal{H}_\eta(r_n)|) = O(r_n^{-2}) = O(n / \ln n). \quad (2)$$

Now pick an arbitrary $\mathcal{S} \in \mathcal{U}$, and let $\mathcal{S}' \subseteq \mathcal{S}$ be the set of those $q \in \mathcal{S}$ for which $q + [0, \eta r_n]^2 \subseteq [0, 1]^2$. Since $\mathcal{S} \subseteq p + (-10^5 r_n, 10^5 r_n)^2$ for any $p \in \mathcal{S}$, we have that $|\mathcal{S} \setminus \mathcal{S}'| \leq 4 \cdot \lceil 2 \cdot 10^5 / \eta \rceil$. And hence, by choice of η_0 , we have

$$|\mathcal{S}'| > (1 + \frac{\varepsilon}{2}) \pi \eta^{-2}. \quad (3)$$

Let $Z := |\{X_1, \dots, X_n\} \cap (\bigcup_{q \in \mathcal{S}'} q + [0, \eta r_n]^2)|$ denote the (random) number of X_i that fall into one of the squares $q + [0, \eta r_n]^2$ with $q \in \mathcal{S}'$. Then $Z \sim \text{Bi}(n, |\mathcal{S}'| \eta^2 r_n^2)$. Appealing to Lemma 8:

$$\begin{aligned} \mathbb{P}[q \text{ is sparse for all } q \in \mathcal{S}] &\leq \mathbb{P}[q \text{ is sparse for all } q \in \mathcal{S}'] \\ &\leq \mathbb{P}[Z \leq 99|\mathcal{S}'|] \\ &\leq \exp[-n|\mathcal{S}'|\eta^2 r_n^2 H(99/\eta^2 n r_n^2)], \end{aligned} \quad (4)$$

where $H(x) = x \ln x - x + 1$. Now notice that, by (3)

$$n|\mathcal{S}'|\eta^2 r_n^2 H(99/\eta^2 n r_n^2) > (1 + \frac{\varepsilon}{2}) \pi n r_n^2 H(99/\eta^2 n r_n^2) = (1 + \frac{\varepsilon}{2} + o(1)) \ln n, \quad (5)$$

where the last equality holds by the choice of $r_n = \sqrt{\ln n / \pi n}$ and the fact that $H(x) \rightarrow 1$ as $x \downarrow 0$ (note $99/(\eta^2 n r_n^2) = O(1/\ln n) \rightarrow 0$). Combining (2), (4) and (5), the union bound now gives us that

$$\begin{aligned} \mathbb{P}[\exists \mathcal{S} \in \mathcal{U} \text{ such that } q \text{ sparse for all } q \in \mathcal{S}] &\leq \sum_{\mathcal{S} \in \mathcal{U}} \mathbb{P}[q \text{ sparse for all } q \in \mathcal{S}] \\ &\leq |\mathcal{U}| n^{-1 - \frac{\varepsilon}{2} + o(1)} \\ &= o(1), \end{aligned}$$

which proves part **(i)**.

Now let $\mathcal{U}_{\text{side}} \subseteq \mathcal{U}$ denote the collection of all $\mathcal{S} \subseteq \mathcal{H}_\eta(r_n)$ that satisfy the conditions of part **(ii)** of the lemma. Noticing that

$$|\mathcal{H}_\eta(r_n) \cap \text{side}(10^5 r_n)| \leq 4 \cdot \lceil 10^5 / \eta \rceil \cdot (1 + 1/\eta r_n) = O(1/r_n) = O(\sqrt{n / \ln n}),$$

and reusing (1), we see that:

$$|\mathcal{U}_{\text{side}}| \leq \sum_{p \in \mathcal{H}_\eta(r) \cap \text{side}(10^5 r_n)} 2^{N(p)} = O(\sqrt{n / \ln n}). \quad (6)$$

Now let $\mathcal{S} \in \mathcal{U}_{\text{side}}$ be arbitrary, and let $\mathcal{S}' \subseteq \mathcal{S}$ be those $q \in \mathcal{S}$ for which $q + [0, \eta r_n]^2 \subseteq [0, 1]^2$. Again we have $|\mathcal{S} \setminus \mathcal{S}'| \leq 4 \lceil 2 \cdot 10^5 \eta^{-1} \rceil < \frac{\varepsilon}{2} \eta^{-2}$, so that this time $|\mathcal{S}'| > (1 + \frac{\varepsilon}{2}) \frac{\pi}{2} \eta^{-2}$. The inequality (4) is still valid and, analogously to (5), we now have $n|\mathcal{S}'|\eta^2 r_n^2 H(99/\eta^2 n r_n^2) > (\frac{1}{2} + \frac{\varepsilon}{4} + o(1)) \ln n$. Combining these observations with (6), the union bound thus gives:

$$\mathbb{P}[\exists \mathcal{S} \in \mathcal{U}_{\text{side}} \text{ such that } q \text{ sparse for all } q \in \mathcal{S}] \leq |\mathcal{U}_{\text{side}}| n^{-\frac{1}{2} - \frac{\varepsilon}{4} + o(1)} = o(1),$$

proving part **(ii)** of the lemma.

Finally, let $\mathcal{U}_{\text{corner}}$ denote the collection of sets $\mathcal{S} \subseteq \mathcal{H}_\eta(r_n)$ that satisfy the conditions of part (iii) of the lemma. Notice that $|\mathcal{H}_\eta(r_n) \cap \text{corner}(10^5 r_n)| \leq 4 \lceil 10^5 \eta^{-1} \rceil^2 = O(1)$. Therefore, also:

$$|\mathcal{U}_{\text{corner}}| \leq \sum_{p \in \mathcal{H}_\eta(r_n) \cap \text{corner}(10^5 r_n)} 2^{N(p)} = O(1). \quad (7)$$

Pick an arbitrary $\mathcal{S} \in \mathcal{U}_{\text{corner}}$ and let $\mathcal{S}' \subseteq \mathcal{S}$ be the set of those $q \in \mathcal{S}$ for which $q + [0, \eta r_n]^2 \subseteq [0, 1]^2$. Then $|\mathcal{S}'| > \frac{\varepsilon}{2} \eta^{-2}$. Again the inequality (4) is still valid and, analogously to (5), we now have $n|\mathcal{S}'| \eta^2 r_n^2 H(99/\eta^2 n r_n^2) > (\frac{\varepsilon}{2} + o(1)) \ln n$. Combining this with (7), the union bound gives:

$$\mathbb{P}[\exists \mathcal{S} \in \mathcal{U}_{\text{corner}} \text{ such that } q \text{ sparse for all } q \in \mathcal{S}] \leq |\mathcal{U}_{\text{corner}}| n^{-\frac{\varepsilon}{2} + o(1)} = o(1).$$

This proves part (iii) of the lemma. \blacksquare

We say that a set $A \subseteq \mathbb{R}^2$ is a *Boolean combination* of the sets $A_1, \dots, A_n \subseteq \mathbb{R}^2$ if A can be constructed from A_1, \dots, A_n by means of any number of compositions of the operations intersection, union and complement. Recall that a *halfplane* is a set of the form $H(a, b) := \{z \in \mathbb{R}^2 : z \cdot a \leq b\}$ for some vector $a \in \mathbb{R}^2 \setminus \{0\}$ and constant $b \in \mathbb{R}$.

Lemma 9. *There exists a constant $C > 0$ such that the following holds for all $\eta, r > 0$. For every $A \subseteq \mathbb{R}^2$ with $\text{diam}(A) < 10^5 r_n$ that is a Boolean combination of at most 1000 halfplanes and balls of radius $\leq r$, we have that $|A \cap \mathcal{H}_\eta(r)| \geq \text{area}(A \cap [0, 1]^2)/(\eta r)^2 - C\eta^{-1}$.*

Proof: Set $C := 10^9$, and let $\eta, r > 0$ be arbitrary. Let $A \subseteq \mathbb{R}^2$ be an arbitrary set that satisfies the two conditions from the lemma. Let

$$A' := \{z \in \mathbb{R}^2 : B(z; \eta r \sqrt{2}) \subseteq A \cap [0, 1]^2\}.$$

In other words, $A' \subseteq A \cap [0, 1]^2$ is the set of all z that are distance at least $\eta r \sqrt{2}$ away from the boundary of $A \cap [0, 1]^2$. Observe that if $q + [0, \eta r]^2$ intersects A' then it is completely contained in A . Because the squares $q + [0, \eta r]^2 : q \in \mathcal{H}_\eta(r)$ are disjoint and cover $[0, 1]^2$ this shows that

$$|A \cap \mathcal{H}_\eta(r)| \geq \text{area}(A')/(\eta r)^2.$$

It thus suffices to bound the area of $A \cap [0, 1]^2 \setminus A'$. The set $A \cap [0, 1]^2$ is also a Boolean combination of halfplanes and balls of radius $\leq r$, this time at most 1004 of them. Let $H_1 = H(a_1, b_1), \dots, H_m = H(a_m, b_m)$ and $B_1 := B(z_1; s_1), \dots, B_k = B(z_k; s_k)$ denote the halfplanes and disks that $A \cap [0, 1]^2$ is a Boolean combination of, where $m + k \leq 1004$ and $s_1, \dots, s_k \leq r$. We can assume w.l.o.g. that $\|a_i\| = 1$ for $i = 1, \dots, m$. Pick an arbitrary $z_0 \in A$. Because $\text{diam}(A) < 10^5 r$ we have $A \subseteq B(z_0, 10^5 r)$. Let us now observe that if $z \in (A \cap [0, 1]^2) \setminus A'$ then z lies within distance $\eta r \sqrt{2}$ of the boundary of one of the sets H_i or one of the B_j . This implies that

$$A \cap [0, 1]^2 \setminus A' \subseteq H'_1 \cup \dots \cup H'_m \cup B'_1 \cup \dots \cup B'_k,$$

where $H'_i := B(z_0, 10^5 r) \cap H(a_i, b_i + \eta r \sqrt{2}) \setminus H(a_i, b_i - \eta r \sqrt{2})$ for $i = 1, \dots, m$ and $B'_j := B(z_j; s_j + \eta r \sqrt{2}) \setminus B(z_j; s_j - \eta r \sqrt{2})$ for $j = 1, \dots, k$. Now notice that $\text{area}(H'_i) \leq (2 \cdot 10^5 r) \times (2\eta r \sqrt{2}) = 4 \cdot 10^5 \eta r^2 \sqrt{2}$ and $\text{area}(B'_j) = \pi((s_j + \eta r \sqrt{2})^2 - (s_j - \eta r \sqrt{2})^2) = 4\pi s_j \eta r \sqrt{2} \leq 4\pi \eta r^2 \sqrt{2}$. Thus

$$\text{area}(A \cap [0, 1]^2 \setminus A') \leq 1004 \cdot 4 \cdot 10^5 \eta r^2 \sqrt{2} \leq C\eta r^2,$$

which gives $|A \cap \mathcal{H}_\eta(r)| \geq (\text{area}(A \cap [0, 1]^2) - C\eta r^2)/(\eta r)^2 = \text{area}(A \cap [0, 1]^2)/(\eta r)^2 - C\eta^{-1}$ as required. \blacksquare

Proof of Proposition 5: Set $V_n = \{X_1, \dots, X_n\}$, $r_n = \sqrt{\ln n / \pi n}$. We will show how Lemma 7 can be applied to show that each of the statements (P1)-(P6) hold with high probability for $\mathcal{H}_\eta(r_n), \mathcal{D}_\eta(V_n, r_n), \mathcal{B}_\eta(V_n, r_n)$ if η is chosen sufficiently small.

Set $\varepsilon := 1/1000$. Fix an $0 < \eta < \eta_0(\varepsilon)$, where η_0 is as in Lemma 7, that is also small enough for the following three inequalities to hold:

$$\begin{aligned} (1 + \frac{1}{100})(1 - \eta\sqrt{2})^2\pi - C\eta &> (1 + \varepsilon)\pi, \\ (1 + \frac{1}{100})(1 - \eta\sqrt{2})^2\frac{\pi}{2} - C\eta &> (1 + \varepsilon)\frac{\pi}{2}, \\ (1 - \eta\sqrt{2})^2\frac{\pi}{4} - C\eta &> \varepsilon, \end{aligned} \tag{8}$$

where C is the constant from Lemma 9.

For any $r > 0$ (and η, ε as chosen above) let $\mathcal{U}(r)$ denote the set of all $\mathcal{S} \subseteq \mathcal{H}_\eta(r)$ for which $\text{diam}(\mathcal{S}) < 10^5 r$ and either $|\mathcal{S}| > (1 + \varepsilon)\pi\eta^{-2}$, or $\mathcal{S} \subseteq \text{side}(10^5 r)$ and $|\mathcal{S}| > (1 + \varepsilon)\frac{\pi}{2}\eta^{-2}$, or $\mathcal{S} \subseteq \text{corner}(10^5 r)$ and $|\mathcal{S}| > \varepsilon\eta^{-2}$. By Lemma 7 it holds with high probability that any $\mathcal{S} \in \mathcal{U}(r_n)$ contains a point that is dense wrt. V_n . To prove the proposition it thus suffices to show that for any $V \subseteq [0, 1]^2$ and $0 < r < 10^{-10}$ that are such that each $\mathcal{S} \in \mathcal{U}(r)$ contains a point that is dense wrt. V the properties **(P1)**-**(P6)** hold for $\mathcal{H}_\eta(r)$, $\mathcal{D}_\eta(V, r)$ and $\mathcal{B}_\eta(V, r)$ (with η, ε as chosen above). Let us thus pick such a $V \subseteq [0, 1]^2$ and $0 < r < 10^{-10}$ for which every $\mathcal{S} \in \mathcal{U}(r)$ contains a point that is dense wrt. V .

Proof that (P1) holds: Aiming for a contradiction, suppose there is some component \mathcal{K} of $\mathcal{D}_\eta(n, r)$ with (geometric) diameter $r' \leq \text{diam}(\mathcal{K}) \leq 1000r$. Let us pick points $p_L, p_R, p_T, p_B \in \mathcal{K}$, where p_L is a point of \mathcal{K} with smallest x -coordinate amongst all points of \mathcal{K} , p_R is a point of \mathcal{K} with biggest x -coordinate, p_B is a point of \mathcal{K} with smallest y -coordinate and p_T is a point of \mathcal{K} with biggest y -coordinate (note these points need not be distinct or unique). See Figure 1 for an illustration. Since $\text{diam}(\mathcal{K}) \geq r'$, we have either $(p_R)_x - (p_L)_x \geq r'/\sqrt{2}$ or $(p_T)_y - (p_B)_y \geq r'/\sqrt{2}$. Without loss of generality, let us assume that $(p_R)_x - (p_L)_x \geq r'/\sqrt{2}$. For $z \in \mathbb{R}^2$ and $s \geq 0$ let us set:

$$\begin{aligned} B_L(z, s) &:= \{z' \in B(z, s) : z'_x < z_x\}, & B_R(z, s) &:= \{z' \in B(z, s) : z'_x > z_x\}, \\ B_B(z, s) &:= \{z' \in B(z, s) : z'_y < z_y\}, & B_T(z, s) &:= \{z' \in B(z, s) : z'_y > z_y\}. \end{aligned}$$

Now define

$$A := B_L(p_L, r') \cup B_R(p_R, r') \cup B_B(p_B, r') \cup B_T(p_T, r'),$$

and let $\mathcal{S} := A \cap \mathcal{H}_\eta(r)$ denote the set of all points of $\mathcal{H}_\eta(r)$ that fall inside A . Let us observe that, since \mathcal{K} is a component of $\mathcal{D}_\eta(n, r)$, the set \mathcal{S} cannot contain any dense q . We also note that $\text{diam}(A) < 10^5 r$ and A is a Boolean combination of ≤ 1000 halfspaces and balls of radius $\leq r$.

Let us define

$$B'_B := \{z \in B_B(p_B, r') : (p_L)_x < z_x < (p_R)_x\}, \quad B'_T := \{z \in B_T(p_T, r') : (p_L)_x < z_x < (p_R)_x\}.$$

Then the sets $B_L(p_L, r'), B_R(p_R, r'), B'_B, B'_T$ are disjoint (see Figure 1). Now observe that, because $(p_R)_x - (p_L)_x > r'/\sqrt{2}$ and $(p_L)_x \leq (p_T)_x \leq (p_R)_x$, the area of B'_T is at least a fraction $(r'/\sqrt{2})/(2r') = 1/2\sqrt{2}$ of the area of $B_T(p_T, r')$. Similarly B'_B has at least $1/2\sqrt{2}$ of the area of $B_B(p_B, r')$. In other words, $\text{area}(B'_B), \text{area}(B'_T) \geq \frac{1}{4\sqrt{2}}\pi(r')^2$, and thus

$$\text{area}(A) \geq (1 + \frac{1}{2\sqrt{2}})\pi(r')^2. \tag{9}$$

First suppose that A is completely contained in $[0, 1]^2$. In this case, Lemma 9 tells us that

$$\begin{aligned} |\mathcal{S}| &\geq \text{area}(A)/(\eta r)^2 - C\eta^{-1} \\ &\geq (1 + \frac{1}{2\sqrt{2}})(1 - \eta\sqrt{2})^2\pi\eta^{-2} - C\eta^{-1} \\ &> (1 + \varepsilon)\pi\eta^{-2}, \end{aligned}$$

where the last inequality holds by (8). We see that $\mathcal{S} \in \mathcal{U}(r)$. But then there must be a dense $q \in \mathcal{S}$, which contradicts that \mathcal{K} is a component of $\mathcal{D}_\eta(V, r)$.

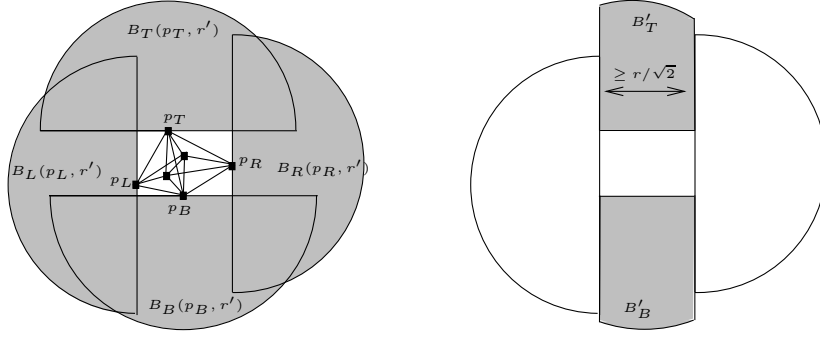


Figure 1: A has area at least $(1 + \frac{1}{2\sqrt{2}})\pi(r')^2$.

Now assume that one of the points p_L, p_R, p_B, p_T is within distance r' of one of the sides of $[0, 1]^2$, but none of these points is an element of $\text{corner}(r')$. Then certainly $\mathcal{S} \subseteq \text{side}(10^5 r)$. Also note that at least one of B'_B, B'_T must be completely contained in $[0, 1]^2$. Moreover, at least half the area of $B_L(p_L, r') \cup B_R(p_R, r')$ lies in $[0, 1]^2$. (If the points are close to $\{0\} \times [0, 1]$ then $B_R(p_R, r') \subseteq [0, 1]^2$. If they are close to $\{1\} \times [0, 1]$ then $B_L(p_L, r') \subseteq [0, 1]^2$. If they are close to $[0, 1] \times \{0\}$ then the top halves of $B_L(p_L, r')$ and $B_R(p_R, r')$ lie completely in $[0, 1]^2$. If they are close to $[0, 1] \times \{1\}$ then the bottom halves of $B_L(p_L, r')$ and $B_R(p_R, r')$ are completely contained in $[0, 1]^2$.) Hence

$$\text{area}(A \cap [0, 1]^2) \geq (1 + \frac{1}{2\sqrt{2}})\frac{\pi}{2}(r')^2.$$

Using Lemma 9 and (8) we find:

$$\begin{aligned} |\mathcal{S}| &\geq \text{area}(A \cap [0, 1]^2)/(\eta r)^2 - C\eta^{-1} \\ &\geq (1 + \frac{1}{2\sqrt{2}})(1 - \eta\sqrt{2})^2 \frac{\pi}{2} \eta^{-2} - C\eta^{-1} \\ &> (1 + \varepsilon) \frac{\pi}{2} \eta^{-2}. \end{aligned}$$

Again we see that $\mathcal{S} \in \mathcal{U}(r)$. So again at least one $q \in \mathcal{S}$ must be dense, which again contradicts that \mathcal{K} was a component of $\mathcal{D}_\eta(V, r)$.

Finally assume that one of the 4 points is an element of $\text{corner}(r')$. Clearly $\mathcal{S} \subseteq \text{corner}(10^5 r)$. Also note that at least one of B'_B, B'_T is completely contained in $A \cap [0, 1]^2$. Hence, by Lemma 9 and (8):

$$\begin{aligned} |\mathcal{S}| &\geq \text{area}(A \cap [0, 1]^2)/(\eta r)^2 - C\eta^{-1} \\ &\geq (1 - \eta\sqrt{2})^2 \frac{1}{4\sqrt{2}} \eta^{-2} - C\eta^{-1} \\ &> \varepsilon \eta^{-2}. \end{aligned}$$

And again this implies $\mathcal{S} \in \mathcal{U}(r)$, which in turn implies the existence of a dense $q \in \mathcal{S}$, which cannot be. We can thus conclude that no component \mathcal{K} of diameter $r' \leq \text{diam}(\mathcal{K}) \leq 1000r$ exists in $\mathcal{D}_\eta(V, r)$.

Proof that (P2) holds: Suppose that $\mathcal{K}_1, \mathcal{K}_2$ are distinct components of $\mathcal{D}_\eta(V, r)$, both with diameter $\leq r'$, such that there exists a point in \mathcal{K}_1 and a point in \mathcal{K}_2 with distance at most $1000r$ between them. Now set $\mathcal{K} := \mathcal{K}_1 \cup \mathcal{K}_2$. Then $r' < \text{diam}(\mathcal{K}) \leq 1002r$ (to see the lower bound, note that any point in \mathcal{K}_1 has distance $> r'$ to any point of \mathcal{K}_2 as they are in distinct components). Let $p_L \in \mathcal{K}$ be a point of smallest x -coordinate, let $p_R \in \mathcal{K}$ be a point of largest x -coordinate, let $p_B \in \mathcal{K}$ be a point of smallest y -coordinate, let $p_T \in \mathcal{K}$ be a point of largest y -coordinate and set $A := B_L(p_L, r') \cup B_R(p_R, r') \cup B_B(p_B, r') \cup B_T(p_T, r')$. Then $\mathcal{S} := A \cap \mathcal{H}_\eta(r)$ cannot contain any

dense point (if, for example, $B_L(p_L, r')$ were to contain a dense point, then this point would lie in the same component of $\mathcal{D}_\eta(V, r)$ as p_L and have a smaller x -coordinate than p_L). We can now proceed as in the proof of **(P1)** to arrive at a contradiction.

Proof that (P3) holds: Suppose that \mathcal{K}_1 is a component of $\mathcal{D}_\eta(V, r)$ with $\text{diam}(\mathcal{K}_1) < r'$ and that $p \in \mathcal{B}_\eta(V, r)$ is a bad point that is at distance $< 1000r$ to some point of \mathcal{K}_1 . Let us set $\mathcal{K} := \mathcal{K}_1 \cup \{p\}$. Then $r' < \text{diam}(\mathcal{K}) < 1001r$ (to see the lower bound, note that any bad point has distance $> r'$ to any dense point). Defining p_L, p_R, p_B, p_T, A and \mathcal{S} as in the proofs of **(P1)** and **(P2)**, we see that \mathcal{S} again cannot contain any dense point. We again arrive at a contradiction by proceeding as in the proof of **(P1)**.

Proof that (P4) holds: Suppose that $p_1, p_2 \in \mathcal{B}_\eta(V, r)$ are bad and that $r' \leq \|p_1 - p_2\| \leq 1000r$. Setting $\mathcal{K} := \{p_1, p_2\}$ and repeating the same argument again gives a contradiction.

Proof that (P5) holds: Suppose there exist $p_1, p_2 \in \mathcal{D}_\eta(n, r)$ with $\|p_1 - p_2\| < 25r$ such that both points are in components of diameter $\geq r'$, and there is no $p_1 p_2$ -path that stays inside $B(p_1, 100r)$. By **(P1)** p_1, p_2 must each be in a component of diameter $> 1000r$. For $k = 25, \dots, 70$ let $S_k := p_1 + [-kr, kr]^2$ denote a square of side length $2kr$ with center p_1 , and let $R_k := S_k \setminus S_{k-1}$ for $k = 26, \dots, 70$. Consider the subgraph $\tilde{\mathcal{D}}$ of $\mathcal{D}_\eta(V, r)$ induced by the points of $\mathcal{D}_\eta(V, r)$ that lie inside S_{70} . Observe that p_1 and p_2 must lie in distinct components $\tilde{\mathcal{K}}_1, \tilde{\mathcal{K}}_2$ of $\tilde{\mathcal{D}}$ (otherwise there is a $p_1 p_2$ -path that stays inside $S_{70} \subseteq B(p_1, 100r)$) and that R_k must contain a point of $\tilde{\mathcal{K}}_1$ and of $\tilde{\mathcal{K}}_2$ for each $k = 26, \dots, 70$ (otherwise, if $\tilde{\mathcal{K}}_j$ misses R_k for $k \geq 26$, then, since $p_j \in S_{25}$, $\tilde{\mathcal{K}}_j$ is also a component of the entire graph $\mathcal{D}_\eta(V, r)$ which has diameter $< 1000r$ and contains p_j , contradicting the earlier observation that the diameter of the component that contains p_j is $> 1000r$). See Figure 2 (the left part). Pick an arbitrary $26 \leq k \leq 70$. Let $q_1 \in \tilde{\mathcal{K}}_1, q_2 \in \tilde{\mathcal{K}}_2$ be two points inside

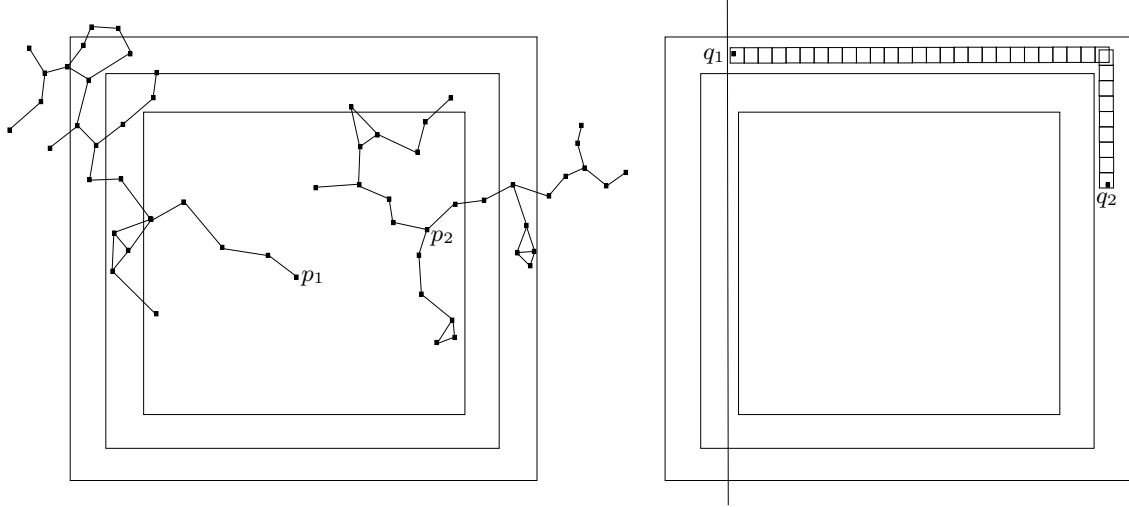


Figure 2: Two points p_1, p_2 at small distance, in large components, but without a short path between them.

R_k . Provided that $(p_1)_x, (p_1)_y \notin ((k-1)r, (k-1+1/2\sqrt{2})r) \cup (1-(k-1+1/2\sqrt{2})r, 1-(k-1)r)$, it is easy to construct a sequence of squares $T_1, \dots, T_m \subseteq R_k \cap [0, 1]^2$, each of side length $r'/2\sqrt{2}$, such that $q_1 \in T_1$ and $q_2 \in T_m$, and $T_i \cap T_{i+1} \neq \emptyset$ for all $i = 1, \dots, m-1$ (see Figure 2, the right part). Observe that every point of T_i is at distance $\leq r'$ of every point of T_{i+1} . Hence, if every T_i were to contain at least one point of $\mathcal{D}_\eta(V, r)$, then there would be a path between q_1 and q_2 in $\tilde{\mathcal{D}}$. But this cannot be since $\tilde{\mathcal{K}}_1$ and $\tilde{\mathcal{K}}_2$ are distinct components of $\tilde{\mathcal{D}}$. Hence, for each $26 \leq k \leq 70$ for which $(p_1)_x, (p_1)_y \notin ((k-1)r, (k-1+1/2\sqrt{2})r) \cup (1-(k-1+1/2\sqrt{2})r, 1-(k-1)r)$ (note there are at most 2 values of k for which this fails), there is at least one square $T \subseteq R_k$ of side length $r'/2\sqrt{2}$ that does not contain any dense point.

For every $26 \leq k \leq 70$ for which this is possible, pick such a square, let A denote the union of

these squares, and set $\mathcal{S} := A \cap \mathcal{H}_\eta(r)$. Clearly $\text{diam}(A) < 10^5 r$ and A is a Boolean combination of less than 1000 halfplanes and balls of radius $\leq r'$. Lemma 9 and (8) thus give:

$$\begin{aligned} |\mathcal{S}| &\geq \text{area}(A)/(\eta r)^2 - C\eta^{-1} \\ &= \frac{43}{8}(1 - \eta\sqrt{2})^2\eta^{-2} - C\eta^{-1} \\ &> (1 + \varepsilon)\pi\eta^{-2}. \end{aligned}$$

But then some $q \in \mathcal{S}$ must be dense, contradiction.

Proof that (P6) holds: Let us call a point $p \in \mathcal{D}_\eta(V, r)$ *large* if it is in a component of diameter $\geq r'$. We first claim that any square $A \subseteq [0, 1]^2$ of side length $5r$ contains a large point. To see this, pick such a square A , remove a vertical strip of width r from the middle, and denote the two remaining rectangles of dimensions $2r \times 5r$ by A_1, A_2 . Let $\mathcal{S}_j := A_j \cap \mathcal{H}_\eta(r)$ for $j = 1, 2$. Then, by Lemma 9 and (8) we have $|\mathcal{S}_j| \geq 10\eta^{-2} - C\eta^{-1} > (1 + \varepsilon)\pi\eta^{-2}$. Hence, each \mathcal{S}_j contains at least one dense point. A dense point in \mathcal{S}_1 and a dense point in \mathcal{S}_2 have distance between r and $5r\sqrt{2}$, so by (P2) at least one of them is large. So the claim holds.

Now pick two arbitrary large points p_1, p_2 of $\mathcal{D}_\eta(V, r)$. It is easy to construct a sequence T_1, \dots, T_m of squares of side $5r$ such that $T_i \subseteq [0, 1]^2$ for $i = 1, \dots, m$, $p_1 \in T_1, p_2 \in T_m$ and $T_i \cap T_{i+1} \neq \emptyset$ for $i = 1, \dots, m-1$. Observe that any point in T_i and any point in T_{i+1} have distance $< 25r$. By (P5) every large point of T_i is in the same component as every large point in T_{i+1} , and, since every T_i has at least one large point, this gives that p_1 and p_2 lie in the same component.

Since p_1, p_2 were arbitrary large points, this shows that all large points lie in the same component. There is at least one large point (inside any square of side $5r$), so that there indeed is exactly one component of diameter $\geq r'$. \blacksquare

It now remains to prove Theorem 6. We will make use of the following observation that is essentially to be found in [5], but is not stated explicitly there. For completeness we include the (short) proof.

Lemma 10. *Any connected unit disk graph has a spanning tree of maximum degree ≤ 26 .*

Proof: Let $G = G(V, r)$ be a connected unit disk graph. For $i, j \in \mathbb{Z}$ set $V_{i,j} := V \cap [ir/\sqrt{2}, (i+1)r/\sqrt{2}) \times [jr/\sqrt{2}, (j+1)r/\sqrt{2})$. Observe that the vertices of $V_{i,j}$ form a clique in G for each i, j , and that there can be an edge vw in G between $w \in V_{i,j}$ and $v \in V_{k,l}$ only if $|i-k|, |j-l| \leq 2$. We construct a subgraph T of G as follows:

- For each i, j such that $V_{i,j} \neq \emptyset$ we delete all edges between distinct vertices of $V_{i,j}$ except for a path going through all its vertices.
- For each pair $(i, j) \neq (k, l)$ such that there exists an edge between a vertex in $V_{i,j}$ and a vertex in $V_{k,l}$ we delete all but one of these edges.

Observe that if vw is an edge of G then there is a vw -path in T . Hence T is a spanning subgraph of G . It is also clear that T has maximum degree at most 26, because any vertex is joined to at most 2 vertices in the same $V_{i,j}$ and at most 24 vertices in different $V_{i,j}$ s. If T is not a tree then we can delete additional edges to make it into a tree. \blacksquare

Proof of Theorem 6: Let $0 < \eta < 1/\sqrt{2}$, $V \subseteq \mathbb{R}^2$ and $r > 0$ be such that $\mathcal{H}_\eta(r), \mathcal{D}_\eta(V, r)$ and $\mathcal{B}_\eta(r)$ satisfy properties (P1)-(P6) and suppose that $r \leq \rho \leq 2r$ is such that $G(V, \rho)$ is 2-connected. Let us enumerate the components of $\mathcal{D}_\eta(V, r)$ and the components of $\mathcal{B}_\eta(V, r)$ as $\mathcal{K}_1, \dots, \mathcal{K}_m$, where \mathcal{K}_1 is the unique component of $\mathcal{D}_\eta(V, r)$ of geometric diameter $\geq 1000r$ and all other \mathcal{K}_i have diameter $< r'$ (observe that by (P4) all components of $\mathcal{B}_\eta(V, r)$ have geometric diameter $< r'$). For $p \in \mathcal{H}_\eta(r)$ denote $V_p := V \cap (p + [0, \eta r]^2)$ and for $i = 1, \dots, m$ let us set $V_{\mathcal{K}_i} := \bigcup_{p \in \mathcal{K}_i} V_p$. Observe that if pq is an edge of $\mathcal{H}_\eta(r)$ then vw is an edge of $G(V, \rho)$ for all $v \in V_p, w \in V_q$ (if $v_1 \in V_{p_1}, v_2 \in V_{p_2}$ with $\|p_1 - p_2\| < r' = r(1 - \eta r\sqrt{2})$ then $\|v_1 - v_2\| \leq$

$\|p_1 - p_2\| + \|(v_1 - p_1) - (v_2 - p_2)\| < r' + \eta r \sqrt{2} = r$). Amongst other things this shows that $V_{\mathcal{K}_i}$ induces a clique in $G(V, \rho)$ for $i = 2, \dots, m$.

Claim 11. For each $i = 2, \dots, m$ for which $|V_{\mathcal{K}_i}| > 0$ there are paths P_1^i, P_2^i in $G(V, \rho)$ such that:

- (i) P_1^i, P_2^i both have one endvertex in $V_{\mathcal{K}_i}$ and one endvertex in $V_{\mathcal{K}_1}$ and all their other vertices in $V \setminus \bigcup_{j=1}^m V_{\mathcal{K}_j}$;
- (ii) P_1^i and P_2^i are vertex-disjoint if $|V_{\mathcal{K}_i}| \geq 2$ and if $V_{\mathcal{K}_i} = \{v\}$ they share only the vertex v but no other vertices;
- (iii) There is a $p \in \mathcal{K}_i$ such that both P_1^i and P_2^i are contained in the disk $B(p, 6r)$.

Proof of Claim 11: If $|V_{\mathcal{K}_i}| \geq 2$ then, since $G(V, \rho)$ is 2-connected, we can pick distinct vertices $a_1, a_2 \in V_{\mathcal{K}_1}$ and distinct $b_1, b_2 \in V_{\mathcal{K}_i}$ and a $a_1 b_1$ -path P_1 and a $a_2 b_2$ -path P_2 such that P_1 and P_2 are vertex-disjoint (exercise 4.2.9 on page 173 of [15]). If $|V_{\mathcal{K}_i}| = \{b_1\}$, then we can pick distinct vertices $a_1, a_2 \in V_{\mathcal{K}_1}$ and a $a_1 b_1$ -path P_1 and a $a_2 b_1$ -path P_2 whose only common vertex is b_1 (exercise 4.2.8 on page 173 of [15]). If $|V_{\mathcal{K}_i}| = 1$ then we set $b_2 = b_1$ in the rest of the proof.

By switching to subpaths if necessary, we can assume that a_j is the only vertex of $V_{\mathcal{K}_1}$ on P_j and b_j is the only vertex of $V_{\mathcal{K}_i}$ on P_j for $j = 1, 2$. Let $p_1, p_2 \in \mathcal{K}_i$ be such that $b_j \in V_{p_j}$ for $j = 1, 2$. We will now show that we can assume that $P_j \subseteq B(p_j; 5r)$ for $j = 1, 2$ (which implies that both are contained in $B(p_1, 6r)$ as $\|p_1 - p_2\| \leq r' < r$).

Suppose that P_1 is not contained in $B(p_1; 5r)$. Write $P_1 = w_0 w_1 w_2 \dots w_k$ where $w_0 = b_1$ and $w_k = a_1$. Let j be the first index such that $\|w_j - p_1\| > 2r$. Observe that

$$\|w_j - p_1\| \leq \|w_{j-1} - p_1\| + \rho \leq 4r.$$

Let $p \in \mathcal{H}_\eta(r)$ be such that $w_j \in V_p$. Since $\|w_j - p_1\| - \|p - w_j\| \leq \|p - p_1\| \leq \|w_j - p_1\| + \|p - w_j\|$ and $\|w_j - p\| < \eta r \sqrt{2}$ we have

$$2r' < \|p - p_1\| < 5r.$$

Depending on whether \mathcal{K}_i is a small component of $\mathcal{D}_\eta(V, r)$ or a component of $\mathcal{B}_\eta(V, r)$, by either **(P3)** or **(P4)** we have that p cannot be bad. Hence there is a dense $q \in \mathcal{D}_\eta(V, r)$ with $\|p - q\| \leq r'$. Observe that $\|q - p_1\| \geq \|p - p_1\| - \|q - p\| > r'$. Either **(P2)** or **(P3)** (depending on whether \mathcal{K}_i is a component of $\mathcal{B}_\eta(V, r)$ or a small component of $\mathcal{D}_\eta(V, r)$) now gives that $q \in \mathcal{K}_1$. Let us pick an $a'_1 \in V_q$ that is distinct from a_2 (since q is dense such a a'_1 certainly exists). Then $w_j a'_1$ is an edge of $G(V, \rho)$, and

$$\|a'_1 - p_1\| \leq \|w_j - p_1\| + \|a'_1 - w_j\| \leq 5r.$$

Hence the path $P'_1 = v_1 w_1 \dots w_j a'_1$ is as required. The same argument shows that we can also assume that $P_2 \subseteq B(p_2, 5r)$. ■

Part **(iii)** of Claim 11 implies the following:

Claim 12. P_j^i and $P_{j'}^{i'}$ are vertex disjoint for all $i \neq i' \in \{2, \dots, m\}$ and $j, j' \in \{1, 2\}$.

Proof of Claim 12: Suppose there exists a common vertex v . By part **(iii)** of Claim 11 there exist $p \in \mathcal{K}_i, p' \in \mathcal{K}_{i'}$ with $\|p - p'\| \leq \|p - v\| + \|p' - v\| < 12r$. But this contradicts either **(P2)**, **(P3)** or **(P4)**, depending on what kind of components $\mathcal{K}_i, \mathcal{K}_{i'}$ are. ■

For $i = 2, \dots, m$, and $j = 1, 2$, let a_j^i denote the endpoint of P_j^i in $V_{\mathcal{K}_1}$ and let b_j^i denote the endpoint of P_j^i in $V_{\mathcal{K}_i}$, and let $p_1^i, p_2^i \in \mathcal{K}_1$ be such that $a_j^i \in V_{p_j^i}$. Since $\|p_1^i - p_2^i\| < 25r$, there is a $p_1^i p_2^i$ -path \mathcal{P}_i in \mathcal{K}_1 such that $\mathcal{P}_i \subseteq B(p_1^i, 100r)$ by **(P5)**.

Claim 13. If $i \neq i'$ then \mathcal{P}_i and $\mathcal{P}_{i'}$ are vertex-disjoint.

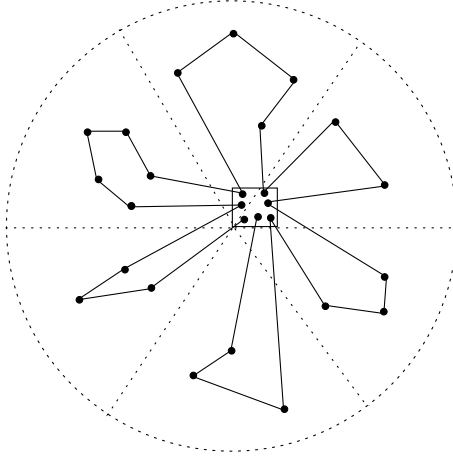


Figure 3: A clean-up path

Proof of Claim 13: Suppose that some $q \in \mathcal{D}(V, r)$ lies on both paths. There is a $p \in \mathcal{K}_i$ such that $\|p - p_1^i\| \leq \|p - a_1^i\| + \|a_1^i - p_1^i\| \leq 6r + \eta r \sqrt{2}$. Hence $\|p - q\| < \|q - p_1^i\| + 7r \leq 107r$. Similarly, there is a $p' \in \mathcal{K}_{i'}$ such that $\|p' - q\| < 107r$. But then $\|p - p'\| < 214r$, which contradicts one of (P2), (P3) or (P4) (depending on what kind of components $\mathcal{K}_i, \mathcal{K}_{i'}$ are). ■

To each vertex $v \in V \setminus \left(\bigcup_{i=1, \dots, m} V_{\mathcal{K}_i} \cup \bigcup_{j=1, 2}^{i=2, \dots, m} P_j^i \right)$ we will attach a label as follows. For such a v there is a $p \in \mathcal{H}_\eta(r)$ such that $v \in V_p$. Note that p cannot be bad (otherwise we would have $v \in V_{\mathcal{K}_i}$ for some i). Hence there is at least one dense $q \in \mathcal{D}_\eta(V, r)$ with $\|p - q\| < r'$. Pick an arbitrary such q and label v with q (note vw is an edge of $G(V, \rho)$ for all $w \in V_q$). For a dense $q \in \mathcal{D}_\eta(V, r)$ let us set $L_q := \{w \in V : w \text{ is labelled } q\}$ and for $i = 1, \dots, m$ will write $L_{\mathcal{K}_i} := \bigcup_{q \in \mathcal{K}_i} L_q$.

Let us observe that for any dense $q \in \mathcal{D}_\eta(V, r)$ and any 7 points $v_1, \dots, v_7 \in V_q$ there is a $v_1 v_7$ -path that contains the vertices of L_q and the vertices v_1, \dots, v_7 but no other vertices. This is because all vertices labelled q are adjacent to all vertices of V_q and the vertices labelled q can be partitioned into 6 cliques, since the vertices labelled q all lie inside the disc $B(q, r')$ and this disk can be dissected into 6 sectors of 60 degrees (each of which has geometric diameter r') – see Figure 3. We will call such a path a *clean-up path* (at q).

Claim 14. For $i = 2, \dots, m$ and each $q \in \mathcal{P}_i$ and every pair $v, w \in V_q$, there exists a vw -path $P_{v,w}^i$ in $G(V, \rho)$ that visits all vertices of $P_1^i, P_2^i, V_{\mathcal{K}_i}$ and $L_{\mathcal{K}_i}$, and at most four vertices from V_p for each $p \in \mathcal{P}_i$, but no other vertices.

Proof of Claim 14: Let us write $\mathcal{P}_i = q_1 \dots, q_N$, where $q = q_j$ for some $1 \leq j \leq N$ and $a_1^i \in V_{q_1}, a_2^i \in V_{q_N}$. We can assume that $v \neq a_2^i$ and $w \neq a_1^i$, by relabelling if necessary.

First suppose that $b_1^i = b_2^i$. In this case we must have $V_{\mathcal{K}_i} = \{b_1^i\}$. But then \mathcal{K}_i must consist of bad points and $L_{\mathcal{K}_i} = \emptyset$. We construct the path $P = P_{v,w}^i$ as follows. Starting from v we go to a vertex $v_{j-1} \in V_{q_{j-1}}$, from there to $v_{j-2} \in V_{q_{j-2}}$ and so on until $v_1 \in V_1$, where we make sure to pick $v_1 = a_1^i$. Next we follow P_1^i to $b_1^i = b_2^i$. (If $j = 1$ and $v = a_1^i$ then we immediately embark on P_1^i . If $j = 1$ and $v \neq a_1^i$ then we first move from v to a_1^i and then embark on P_1^i). Now we follow P_2^i to a_2^i . If it happens that $j = N$ and $w = a_2^i$ then we are done. If $j = N$ and $w \neq a_2^i$ we jump from a_2^i to w and we are done. Otherwise we move from a_2^i to a $v_{N-1} \in V_{q_{N-1}}$, from there to a $v_{N-2} \in V_{q_{N-2}}$ and so on until $v_j \in V_{q_j}$, where we make sure to pick $v_j = w$.

Now assume $b_1^i \neq b_2^i$. For each $q \in \mathcal{K}_i$ such that $|L_q| > 0$, we pick 7 vertices in V_q different from b_1^i, b_2^i (there exist 7 such vertices, because q occurs as a label and is therefore dense) and construct the corresponding clean-up path. We now construct the path $P_{v,w}^i$ as follows. We start

by going from v to b_1^i in the same way as above. Since $V_{\mathcal{K}_i}$ is a clique, we can start from b_1^i , jump to an endvertex of the first clean-up path, follow it, jump from its other endvertex to an endvertex of another clean-up path, follow that path and so on until the last clean-up path. We then follow a path through the remaining vertices of $V_{\mathcal{K}_i}$, arriving at b_2^i . Finally we follow P_2^i to a_2^i , and go from a_2^i back to w in the same way as above. ■

By Lemma 10 there exists a spanning tree \mathcal{T} of \mathcal{K}_1 with maximum degree at most 26. Let $\mathcal{W} = q_0 \dots q_N$ (with $q_0 = q_N$) be a closed walk on \mathcal{T} that traverses every edge exactly twice (once in each direction). Such a walk can for instance be obtained by tracing the steps of a depth-first search algorithm on \mathcal{T} . Observe that \mathcal{W} visits each node $q \in \mathcal{K}_1$ at most 26 times, since the maximum degree of \mathcal{T} is at most 26. We shall now describe a construction of a Hamilton cycle in $G(V, \rho)$. It is convenient to consider "timesteps" $t = 0, \dots, N$, where we envisage ourselves performing the walk \mathcal{W} while at the same time constructing the cycle C . At the beginning of timestep t , the cycle C under construction is always at a vertex $v \in V_{q_t}$ and at the end of timestep $t < N$ we are at a vertex $w \in V_{q_{t+1}}$. We start the cycle from an arbitrary vertex $\alpha_0 \in V_{q_0}$. At the beginning of timestep t we are in some vertex $v \in V_{q_t}$. We now apply the following rules at each timestep $t = 0, \dots, N$:

- Rule 1** If it is the first time \mathcal{W} visits q_t (i.e. q_t is distinct from q_0, \dots, q_{t-1}), and q_t lies on \mathcal{P}_i for some $i = 2, \dots, m$, and it is the first vertex of \mathcal{P}_i that occurs on \mathcal{W} then we pick an arbitrary $w \in V_{q_t} \setminus \{v\}$ and we follow the path $P_{v,w}^i$ from v to w . This timestep has not finished yet. We next apply either Rule 2 or Rule 3 (whichever applies).
- Rule 2** If it is not the last time \mathcal{W} visits q_t , then we simply pick a not yet visited $w \in V_{q_{t+1}}$ and go there. **End of timestep t .**
- Rule 3** If it is the last time that \mathcal{W} visits q_t (i.e. q_t is distinct from q_{t+1}, \dots, q_N) then we do the following. We are currently in a vertex $v \in V_{q_t}$. Pick vertices $v_1, \dots, v_6 \in V_{q_t}$ that have not been visited yet, and follow a clean-up path between v and v_6 that visits all vertices labelled q_t and v_1, \dots, v_6 , but no other vertices. Now we continue by visiting all vertices of V_{q_t} that have not yet been visited. Finally, provided $t < N$, we pick a not yet visited $w \in V_{q_{t+1}}$ and go to w . If $t = N$, then we go to the initial vertex α_0 , completing the cycle. **End of timestep t .**

Let us now explain why this construction works. At each timestep t the Rules 1-3 require unused vertices in V_{q_t} , so we need to argue amongst other things that we never run out of vertices. Pick an arbitrary $q \in \mathcal{K}_1$. By Claim 13 there is at most one $2 \leq i \leq m$ such that $q \in \mathcal{P}_i$. Rule 1 is applied exactly once to a $p \in \mathcal{P}_i$, and when that happens at most 4 new vertices of V_q are used. Rule 2 is applied at most 25 times to q , and each time one new vertex of V_q is used. Thus, at the start of the timestep when \mathcal{W} visits q for the last time, at least $100 - 4 - 25 = 71$ vertices of V_q are left, which is more than enough to construct the clean-up path. So we never get stuck.

We still need to argue that our construction produces a Hamilton cycle. Recall that V can be partitioned into the sets $V_{\mathcal{K}_i}, L_{\mathcal{K}_i} : i = 1, \dots, m$ and $P_j^i \setminus \{a_j^i, b_j^i\} : i = 2, \dots, m, j = 1, 2$ (by construction of $V_{\mathcal{K}_i}$ and $L_{\mathcal{K}_i}$ and by Claim 12). Consider an arbitrary $v \in V$. If $v \in P_j^i \setminus \{a_j^i, b_j^i\}$ for some $i = 2, \dots, m, j = 1, 2$, then C visits v exactly once, namely in the time step when \mathcal{W} first visits a vertex of \mathcal{P}_i . Similarly, if $v \in V_{\mathcal{K}_i}$ or if $v \in L_{\mathcal{K}_i}$ for some $i = 2, \dots, m$, then C visits v exactly once, namely in the time step when \mathcal{W} first visits a vertex of \mathcal{P}_i . If $v \in L_{\mathcal{K}_1}$ then C visits it exactly once, namely at the timestep when \mathcal{W} visits q for the last time where $q \in \mathcal{K}_1$ is such that v is labelled q . It is also clear that C visits every vertex $v \in V_{\mathcal{K}_1}$ exactly once (in Rules 1 and 2 we always take new vertices from V_q , and when Rule 3 is finally applied to q we make sure to visit all remaining vertices of V_q). Thus, C visits every $v \in V$ exactly once and, since in the very end we reconnect to the initial vertex α_0 , it is a Hamilton cycle as required. ■

3 Extension to other norms and higher dimensions

In this section we shall briefly sketch the changes needed to make proof of Theorem 1 work in the case when X_1, X_2, \dots are independent, uniform random points from $[0, 1]^d$ with $d \geq 2$ arbitrary and when $\|\cdot\|$ in the definition of the random geometric graph is the l_p -norm for some $1 < p \leq \infty$. That is,

Theorem 15. *For any $d \geq 2$ and $1 < p \leq \infty$ the following holds. If we pick $X_1, \dots, X_n \in [0, 1]^d$ i.i.d. uniformly at random and we add the edges $X_i X_j$ by order of increasing l_p norm of $X_i - X_j$ then, with probability tending to 1 as $n \rightarrow \infty$, the resulting graph gets its first Hamilton cycle at precisely the same time it loses its last vertex of degree less than two.*

We should perhaps remark that the restriction to the l_p -norm with $1 < p \leq \infty$ is needed only because it is imposed by the results of Penrose that we invoke in our proofs (cf. Theorem 8.4 and 13.17 of [12]). These results of Penrose show a notable difference between the case when the points X_1, \dots, X_n are chosen uniformly at random from the unit hypercube and the case when they are chosen from the d -dimensional torus (i.e. if we identify opposite facets of the unit hypercube). The restriction to l_p -norms with $1 < p \leq \infty$ is imposed by Penrose for the unit hypercube (but not for the torus) to deal with the technical difficulties that arise from “boundary effects”.

Most of the proofs go through almost unaltered if we change the relevant constants etc. in the following way. When a square appears in the proofs for the 2-dimensional, Euclidean case, it should usually be replaced by a d -th power. Instead of $\text{area}(\cdot)$ we need to put $\text{vol}(\cdot)$, the d -dimensional volume. Whenever the constant π occurs it should be replaced by $\theta := \text{vol}(B(0, 1))$, the volume of the unit ball wrt. the l_p -norm. Wherever the constant $\sqrt{2}$ appears, it should be replaced by $d^{1/p} = \text{diam}([0, 1]^d)$, the diameter of the d -dimensional hypercube as measured by the l_p -norm (here we interpret $1/\infty$ as 0, so that $d^{1/\infty} = 1$). For example, we now set $r' := r(1 - \eta d^{1/p})$. Instead of the numbers $10^5, 1000, 100, 25$ we put suitably chosen large constants. In particular, in the definition of $\mathcal{H}_\eta(r), \mathcal{D}_\eta(V, r), \mathcal{B}_\eta(V, r)$ a point of $p \in \mathcal{H}_\eta(r)$ is dense if the cube $p + [0, \eta r]^d$ contains at least K points for a suitably chosen constant K (that will have to be larger than 100 for some choices of d, p).

In the statement and proof of (the analogues of) Proposition 5 and Lemma 7 for the general case we can put

$$r_n := \left((1 - \delta) \left(\frac{2^{d-1}}{d\theta} \right) \ln n / n \right)^{1/d},$$

where $\delta = \delta(\varepsilon)$ is a suitably chosen small constant. It can be read off from Theorem 8.4 in [12], together with Theorem 13.17 in [12] (the version of Theorem 3 for arbitrary dimension and the l_p -norm), that $r_n < \rho_n(2\text{-connected}) < 2r_n$. We should perhaps remark that, although it is possible to have δ tend to 0 in a suitable way, it cannot be disposed of altogether. This is because (c.f. Theorem 8.4 in [12]) the last vertex of degree < 2 disappears when $r = \left(\left(\frac{2^{d-1}}{d\theta} \ln n + c_{d,p} \ln \ln n + O(1) \right) / n \right)^{\frac{1}{d}}$ where $c_{d,p}$ is a constant that is negative for some choices of d, p .

In the higher-dimensional analogue of Lemma 7 we need to distinguish additional subcases to deal with the situation when \mathcal{S} is close to a k -dimensional face of $[0, 1]^d$, for $k = 1, \dots, d-1$. Let $\text{side}_k(s)$ denote the set of all $z \in [0, 1]^d$ that have k coordinates in $[0, s) \cup (1-s, 1]$. Then $|\mathcal{H}_\eta(r_n) \cap \text{side}_k(Kr_n)| = O(r_n^{-(d-k)})$, and also $|\mathcal{U}_k| = O(r_n^{-(d-k)}) = O((n/\ln n)^{(d-k)/d})$, where \mathcal{U}_k is the collection of all sets $\mathcal{S} \subseteq \mathcal{H}_\eta(r) \cap \text{side}_k(Kr)$ with diameter at most Kr . The argument in the proof of Lemma 7 thus shows that each $\mathcal{S} \in \mathcal{U}_k$ with $|\mathcal{S}| > (1 + \varepsilon)^{\frac{d-k}{2^{d-1}}} \theta \eta^{-d}$ contains a dense point.

Lemma 9 and its proof essentially go through unaltered if we replace $\text{area}(\cdot)$ by the d -dimensional volume $\text{vol}(\cdot)$, $\sqrt{2}$ by $d^{1/p}$, η^{-2} by η^{-d} and η^{-1} by $\eta^{-(d-1)}$.

In the proof of **(P1)**, we now pick $2d$ vectors from \mathcal{K} , two for each coordinate. For $i = 1, \dots, d$ we let p_i^- resp. p_i^+ be a point of smallest resp. largest i -th coordinate. We set $A := \bigcup_i B_i^-(p_i^-, r') \cup B_i^+(p_i^+, r')$, where $B_i^-(z, s) := \{z' \in B(z, s) : (z')_i < z_i\}$, $B_i^+(z, s) := \{z' \in B(z, s) : (z')_i > z_i\}$,

and we set $\mathcal{S} := A \cap \mathcal{H}_\eta(r_n)$. Again it is clear that \mathcal{S} cannot contain any dense point if \mathcal{K} is a component. This time there must exist an $1 \leq j \leq d$ such that $p_j^+ - p_j^- > r'/d^{1/p}$. We now need to consider the case when one of these $2d$ points is in $\text{side}_k(r')$ but none lies in $\text{side}_{k+1}(r')$. Wlog. suppose the points are close to the face $\{z \in [0, 1]^d : z_1 = \dots = z_k = 0\}$. First suppose that $j \leq k$. Then we can assume (w.l.o.g.) that $j = 1$. We see that $A \cap [0, 1]^d$ contains A_1 and A_2 , where

$$\begin{aligned} A_1 &:= \{z \in B(p_1^+, r') : z_i > (p_1^+)_i \text{ for } i = 1, \dots, k\}, \quad \text{and} \\ A_2 &:= \{z \in B(p_2^+, r') : (p_1^-)_1 < z_1 < (p_1^+)_1, \text{ and } z_i > (p_2^+)_i \text{ for } i = 2, \dots, k\}. \end{aligned}$$

Note that A_1 and A_2 are disjoint, that $\text{vol}(A_1) = \theta(r')^d/2^k$ and that $\text{vol}(A_2) > \theta(r')^d/2^k d^{1/p}$. Hence

$$\begin{aligned} |\mathcal{S}| &\geq \text{vol}(A \cap [0, 1]^d)/(\eta r_n)^d - C\eta^{-(d-1)} \\ &\geq (1 + 1/d^{1/p})(1 - \eta d^{1/p})^d \theta \eta^{-d}/2^k - C\eta^{-(d-1)} \\ &> (1 + \varepsilon) \theta \eta^{-d}/2^{-k} \\ &\geq (1 + \varepsilon) \frac{d-k}{2^{d-1}} \theta \eta^{-d}, \end{aligned}$$

(provided ε, η were chosen appropriately) so that \mathcal{S} must contain a dense point.

Now consider the case when $j > k$. Then $A \cap [0, 1]^d$ contains

$$\begin{aligned} A_1 &:= \{z \in B_j^+(p_j^+, r') : z_i > (p_j^+)_i \text{ for } i = 1, \dots, k\}, \\ A_2 &:= \{z \in B_j^-(p_j^-, r') : z_i > (p_j^+)_i \text{ for } i = 1, \dots, k\}, \quad \text{and} \\ A_3 &:= \{z \in B(p_1^+, r') : (p_j^-)_j < z_j < (p_j^+)_j, \text{ and } z_i > (p_1^+)_i \text{ for } i = 1, \dots, k\}. \end{aligned}$$

Now A_1, A_2 both have volume $\theta(r')^d/2^{k+1}$ and A_3 has volume at least $\theta(r')^d/2^k d^{1/p}$. So again \mathcal{S} must contain a dense point.

The arguments that reduce **(P2)**-**(P4)** to the proof of **(P1)** work in the same way for other dimensions and norms. In the proof of **(P5)** we now show that (with $C_1 < C_2$ suitable constants) if two points p_1, p_2 have distance less than C_1 and there is no path between them that stays inside $p_1 + [-C_2 r, C_2 r]^d$, then in all but d of the sets $p_1 + [-kr, kr]^d \setminus [-(k-1)r, (k-1)r]^d : k = C_1 + 1, \dots, C_2$ there is a cube of side $1/2d^{1/p}$ without a dense point inside it.

In the proof of **(P6)** we merely need to replace squares of side $5r$ with hypercubes of side Kr for some suitable constant K .

Lemma 10 and its proof generalise to give that any connected, (non-random) geometric graph in dimension d with the l_p -norm has a spanning tree of maximum degree at most $(2\lceil d^{1/p} \rceil + 1)^d + 1$.

The proof of Theorem 6 also generalises with only minor modifications, the most important one being in the definition of a clean-up path. For any dimension d and $1 < p \leq \infty$ there exists a finite k such the unit ball wrt. the l_p norm can be partitioned into k parts each of diameter ≤ 1 (covering the ball by hypercubes of side $1/d^{1/p}$ shows for instance that we can take $k = (2\lceil d^{1/p} \rceil)^d$.) We can thus construct clean-up paths at each $q \in \mathcal{D}_\eta(V, r)$ that use $k + 1$ vertices from V_q .

4 Concluding remarks

In this paper we have shown that, with high probability, the least r for which the random geometric graph $G(n, r)$ is Hamiltonian coincides with the least r for which it has minimum degree at least 2. Recall that a graph is *pancyclic* if it has cycles of all lengths $3 \leq k \leq n$. As shown by Łuczak [9], the usual random graph becomes pancyclic at exactly the same time it loses its last vertex of degree < 2 . It is natural to ask whether a similar statement can be shown for the random geometric graph. As it happens, the answer is yes. Our proof of Theorem 1 can be adapted to show:

Theorem 16. $\mathbb{P}[\rho_n(\text{pancyclic}) = \rho_n(\text{minimum degree} \geq 2)] \rightarrow 1$ as $n \rightarrow \infty$.

This also implies that in Corollary 2 we can replace the word "Hamiltonian" with "pancyclic". Let us briefly explain how to adapt our proof of Theorem 1 to give Theorem 16. First note that the proof would still have gone through if we had defined $p \in \mathcal{H}_\eta(r)$ to be dense if the corresponding square contains at least 1000 points instead of 100. We will reconsider the way we constructed the Hamilton cycle C , and show that for every $1 \leq k \leq n - 3$ there are k points from V that we can omit and construct a cycle through the remaining points applying Rules 1-3 in the same way as in the proof of Theorem 6. For any set of vertices $A \subseteq \bigcup_{i=1}^m L_{\mathcal{K}_i}$ we can construct a cycle through $V \setminus A$ by simply omitting the vertices of A and proceeding as in the proof of Theorem 6 (the vertices of A will simply be omitted from the corresponding clean up paths). Thus we have cycles of lengths $n - k$ for $k = 0, \dots, \sum_i |L_{\mathcal{K}_i}|$. Let us now omit all vertices in $\bigcup_{i \geq 1} L_{\mathcal{K}_i}$, and consider \mathcal{K}_2 . Since \mathcal{K}_2 is a clique, we can omit all vertices except b_1^2, b_2^2 one by one and each time construct a cycle through the remaining points. Let us omit $V_{\mathcal{K}_2} \setminus \{b_1^2, b_2^2\}$ as well as $\bigcup_{i \geq 1} L_{\mathcal{K}_i}$ in the sequel. We can assume w.l.o.g. that P_1^2, P_2^2 have no shortcuts (in other words we can assume they are induced paths), because we could have easily insisted on this in the proof of Claim 11. This means that each of these paths is the union of two stable sets. Notice that if $\{v_1, \dots, v_l\} \subseteq B(p, 6r)$ is a stable set then the discs $B(v_1, r/2), \dots, B(v_l, r/2)$ are disjoint and contained in $B(p, 6\frac{1}{2})$, so that $l \leq \pi(6\frac{1}{2})^2 / \pi(\frac{1}{2})^2 = 169$. This shows that P_1^2, P_2^2 each have at most 338 vertices. For each $k = 1, \dots, |P_1^2| + |P_2^2| - 1$ we can omit k points from V_p for some $p \in \mathcal{P}_2$ (since $|V_p| \geq 1000 > 2 \cdot 338$ this can be done). There is always a cycle through the remaining vertices. Now put back those points from V_p and remove P_1^2 and P_2^2 . Again there is a cycle through all points that have not been removed. Having removed $V_{\mathcal{K}_2}$ and P_1^2, P_2^2 , we can repeat the same procedure for $i = 3, \dots, m$. We see that there is a cycle of length $n - k$ for $k = 0, \dots, \sum_{i=1}^m |L_{\mathcal{K}_i}| + \sum_{i=2}^m |V_{\mathcal{K}_i}| + \sum_{i=2}^m (|P_1^i| + |P_2^i|)$. Removing $\bigcup_{i=1}^m L_{\mathcal{K}_i}$, $\bigcup_{i=2}^m V_{\mathcal{K}_i}$ and $\bigcup_{i=2}^m P_1^i \cup P_2^i$, we are only left with vertices of $V_{\mathcal{K}_1}$, and for each $p \in \mathcal{K}_1$ we still have at least $1000 - 2 = 998$ points of V_p left over (it contains at most 2 endpoints of P_j^i s). We omit the remaining points one by one, starting with points in squares corresponding to leafs of \mathcal{T} . Once we have run out of those, we continue with points in squares corresponding to leafs of the subtree of \mathcal{T} induced by the nonempty squares, and so on. We see that are indeed able to construct cycles of all lengths.

Observe that if $\delta(G)$ denotes the minimum degree of the graph G , then there can be at most $\lfloor \delta(G)/2 \rfloor$ edge disjoint Hamilton cycles in G . Bollobás and Frieze [4] have shown that, with high probability, the ordinary random graph has k edge-disjoint Hamilton cycles for the first time at precisely the same moment it first achieves minimum degree $2k$. Perhaps methods similar to ours will prove:

Conjecture 17. $\rho_n(\text{there exist } k \text{ edge disjoint Hamilton cycles}) = \rho_n(\text{minimum degree} \geq 2k)$ w.h.p., for any fixed $k \in \mathbb{N}$.

Let H_δ denote the graph property that there are $\lfloor \delta(G)/2 \rfloor$ edge disjoint Hamilton cycles in the graph G . It has been conjectured (see e.g. [6]) that H_δ holds w.h.p. for all choices of the sequence $(m_n)_n$ in the $G(n, m_n)$ model. This is known to be true for choices of $(m_n)_n$ for which $G(n, m_n)$ has minimum degree $o(\ln n)$ w.h.p. (c.f. [7]), but it is still open in general. A natural question is therefore:

Question 18. Does H_δ hold w.h.p. for the random geometric graph $G(n, r_n)$ for all choices of the sequence $(r_n)_n$?

References

- [1] M. Ajtai, J. Komlós, and E. Szemerédi. First occurrence of Hamilton cycles in random graphs. In *Cycles in graphs (Burnaby, B.C., 1982)*, volume 115 of *North-Holland Math. Stud.*, pages 173–178. North-Holland, Amsterdam, 1985.
- [2] J. Balogh, B. Bollobás, and M. Walters. Hamilton cycles in random geometric graphs. Preprint, <http://arxiv.org/abs/0905.4650>.

- [3] B. Bollobás. The evolution of sparse graphs. In *Graph theory and combinatorics (Cambridge, 1983)*, pages 35–57. Academic Press, London, 1984.
- [4] B. Bollobás and A. M. Frieze. On matchings and Hamiltonian cycles in random graphs. In *Random graphs '83 (Poznań, 1983)*, volume 118 of *North-Holland Math. Stud.*, pages 23–46. North-Holland, Amsterdam, 1985.
- [5] J. Díaz, D. Mitsche, and X. Pérez. Sharp threshold for Hamiltonicity of random geometric graphs. *SIAM J. Discrete Math.*, 21(1):57–65, 2007.
- [6] A. M. Frieze and M. Krivelevich. On packing Hamilton cycles in ϵ -regular graphs. *J. Combin. Theory Ser. B*, 94(1):159–172, 2005.
- [7] A. M. Frieze and M. Krivelevich. On two Hamilton cycle problems in random graphs. *Israel J. Math.*, 166:221–234, 2008.
- [8] J. Komlós and E. Szemerédi. Limit distribution for the existence of Hamiltonian cycles in a random graph. *Discrete Math.*, 43(1):55–63, 1983.
- [9] T. Łuczak. Cycles in random graphs. *Discrete Math.*, 98(3):231–236, 1991.
- [10] R. Martin. Personal communication.
- [11] M. D. Penrose. On k -connectivity for a geometric random graph. *Random Structures Algorithms*, 15(2):145–164, 1999.
- [12] M. D. Penrose. *Random Geometric Graphs*. Oxford University Press, Oxford, 2003.
- [13] X. Pérez. Personal communication.
- [14] J. Petit. Layout problems. PhD Thesis, Universitat Politècnica de Catalunya, 2001.
- [15] D. B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.